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Globally solvable systems of complex vector fields ☆

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ABSTRACT

We consider a class of involutive systems of n smooth vector fields on the $n+1$ dimensional torus. We obtain a complete characterization for the global solvability of this class in terms of Liouville forms and of the connectedness of all sublevel and superlevel sets of the primitive of a certain 1-form in the minimal covering space.

1. Introduction

In this work we study the global solvability of a system of complex vector fields on the torus $\mathbb{T}^{n+1} \simeq (\mathbb{R}/2\pi\mathbb{Z})^{n+1}$ given by

$$L_j = \frac{\partial}{\partial t_j} + \bigl(a_j(t) + ib_j(t_j)\bigr)\frac{\partial}{\partial x}, \quad j = 1, \ldots, n, \tag{1.1}$$

where $a_j \in C^\infty(\mathbb{T}^n; \mathbb{R})$, $b_j \in C^\infty(\mathbb{T}^1; \mathbb{R})$ and $(t, x) = (t_1, \ldots, t_n, x)$ are the coordinates on the torus $\mathbb{T}^{n+1}$. We assume that the system (1.1) is involutive or equivalently that the 1-form $c = a + ib \in \bigwedge^1 C^\infty(\mathbb{T}^n)$ is closed, where $a(t) = \sum_{j=1}^n a_j(t)\,dt_j$ and $b(t) = \sum_{j=1}^n b_j(t_j)\,dt_j$ are real 1-forms on $\mathbb{T}^n$. For further explanations about these concepts see the books of Berhanu, Cordaro, and Hounie [8] and Treves [12].

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The system (1.1) gives rise to a complex of differential operators $\mathbb{L}$ which at the first level acts in the following way

$$\mathbb{L}u = d_t u + c(t) \wedge \frac{\partial}{\partial x} u, \quad u \in C^\infty\bigl(\mathbb{T}^{n+1}\bigr) \ \bigl(\text{or } \mathcal{D}'\bigl(\mathbb{T}^{n+1}\bigr)\bigr),$$

where d_t denotes the exterior differential on the torus $\mathbb{T}^n_t$.

Note that if $\mathbb{L}u = f$ for some $u \in C^\infty(\mathbb{T}^{n+1})$ (or $\mathcal{D}'(\mathbb{T}^{n+1})$) then f must be of the form

$$f = \sum_{j=1}^{n} f_j(t,x)\, dt_j.$$

As in [3] we denote the space of such forms by $C^\infty(\mathbb{T}^n_t \times \mathbb{T}^1_x; \bigwedge^{1,0})$.

The local solvability of this complex was treated by Treves in his seminal work [11].

Our aim is to carry out a study of the global solvability at the first level of this complex. The real case, that is, when each function b_j is identically zero, was studied by Bergamasco and Petronilho in [7]. When $c(t)$ is exact the problem was solved by Cardoso and Hounie in [9]. The case of a single vector field was treated by Hounie in [10]. We are interested in global solvability when at least one of the functions b_j is not identically zero.

We prove that the global solvability is closely related to the property of all the sublevel and superlevel sets of a global primitive of the pull-back $\Pi^* b$ being connected in the minimal covering space $\Pi : \mathcal{T} \to \mathbb{T}^n$ on which $\Pi^* b$ is exact.

When some of the functions b_j are identically zero the global solvability involves Liouville forms.

An interesting consequence of our results is that the system

$$\begin{cases} L_1 = \dfrac{\partial}{\partial t_1} + \dfrac{1}{4}\dfrac{\partial}{\partial x}, \\ L_2 = \dfrac{\partial}{\partial t_2} + \left(\dfrac{1}{2} + i \sin t_2\right)\dfrac{\partial}{\partial x} \end{cases}$$

is globally solvable on $\mathbb{T}^3$, while

$$\begin{cases} L_1 = \dfrac{\partial}{\partial t_1} + \dfrac{1}{2}\dfrac{\partial}{\partial x}, \\ L_2 = \dfrac{\partial}{\partial t_2} + \left(\dfrac{1}{4} + i \sin t_2\right)\dfrac{\partial}{\partial x} \end{cases}$$

is not.

The articles [1–6] deal with similar questions.

2. Preliminaries and statement of the main result

We study the global solvability of the equation $\mathbb{L}u = f$ where $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ and $f \in C^\infty(\mathbb{T}^n_x \times \mathbb{T}^1_t; \bigwedge^{1,0})$. There are natural compatibility conditions on f for the existence of a solution u to the equation $\mathbb{L}u = f$. We now move on to describing them.

If $f \in C^\infty(\mathbb{T}^n_x \times \mathbb{T}^1_t; \bigwedge^{1,0})$ we consider the Fourier series

$$f(t,x) = \sum_{\xi \in \mathbb{Z}} \hat{f}(t,\xi) e^{i\xi x},$$

where $\hat{f}(t,\xi) = \sum_{j=1}^n \hat{f}_j(t,\xi)\, dt_j$ and $\hat{f}_j(t,\xi)$ denotes the Fourier transform with respect to x.

We may write $c = c_0 + d_t\mathcal{C}$ where $\mathcal{C}$ is a complex valued smooth function of $t \in \mathbb{T}^n$ and $c_0 \in \bigwedge^1 \mathbb{C}^n \simeq \mathbb{C}^n$. Thus, if $f \in C^\infty(\mathbb{T}^n_x \times \mathbb{T}^1_t; \bigwedge^{1,0})$ and if there exists $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ such that $\mathbb{L}u = f$, then, since $\mathbb{L}$ defines a differential complex, $\mathbb{L}f = 0$; also for any $\xi \in \mathbb{Z}$,

$$\hat{f}(t,\xi)e^{i(\psi_\xi(t)+\xi\mathcal{C}(t))} \quad \text{is exact when } \xi c_0 \text{ is integral,} \tag{2.1}$$

where $\psi_\xi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is such that $d\psi_\xi = \Pi^*(\xi c_0)$ and $\Pi : \mathbb{R}^n \to \mathbb{T}^n$ denotes the universal covering of $\mathbb{T}^n$.

We define the following set

$$\mathbb{E} = \left\{ f \in C^\infty\left(\mathbb{T}^n_x \times \mathbb{T}^1_t; \bigwedge^{1,0}\right);\ \mathbb{L}f = 0 \text{ and (2.1) holds} \right\}.$$

Definition 2.1. The operator $\mathbb{L}$ is said to be globally solvable at the first level of the complex on $\mathbb{T}^{n+1}$ if for each $f \in \mathbb{E}$ there exists $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ satisfying $\mathbb{L}u = f$.

We will identify the 1-form $c_0 \in \bigwedge^1 \mathbb{C}^n$ with the vector $(c_{10}, \ldots, c_{n0})$ in $\mathbb{C}^n$ consisting of the periods $c_{j0} = a_{j0} + ib_{j0}$ where

$$a_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(0,\ldots,\tau_j,\ldots,0)\,d\tau_j \quad \text{and} \quad b_{j0} = \frac{1}{2\pi}\int_0^{2\pi} b_j(\tau_j)\,d\tau_j.$$

Also, we will use the notation $a_0 = (a_{10}, \ldots, a_{n0})$ and $b_0 = (b_{10}, \ldots, b_{n0})$.

In [5] the authors define the *minimal* covering of $\mathbb{T}^n$ with respect to the 1-form b as the smallest covering space $\Pi : \mathcal{T} \to \mathbb{T}^n$ where the pull-back Π^*b is exact. The minimal covering of $\mathbb{T}^n$ is isomorphic to $\mathcal{T} = \mathbb{R}^r \times \mathbb{T}^{n-r}$ where r is the rank of the group of the periods of b. The extreme cases are $\mathcal{T} = \mathbb{T}^n$ when $r = 0$ and $\mathcal{T} = \mathbb{R}^n$ when $r = n$. Also, note that b is exact if and only if $r = 0$.

In the minimal covering space the 1-form Π^*b has a global primitive B and since each b_j depends only on the coordinate t_j the function $B : \mathcal{T} \to \mathbb{R}$ is of the form $B(t) = \sum_{j=1}^n B_j(t_j)$.

Given $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Q}^N$ and $J = (j_1, \ldots, j_m) \in \mathbb{N}^m$ we denote by q_J the smallest positive integer such that $q_J(\alpha_{j_1}, \ldots, \alpha_{j_m}) \in \mathbb{Z}^m$. When $J = (1, \ldots, N)$ we write $q_* \doteq q_J$; thus $q_J \leqslant q_*$ and q_J divides q_*.

If, otherwise, $\alpha \notin \mathbb{Q}^N$ we say that α is *Liouville* when there exist a constant $C > 0$ and a sequence $\{(\kappa_l, \xi_l)\}$ in $\mathbb{Z}^N \times \mathbb{Z}$ ($\xi_l \geqslant 2$) such that

$$\max_{j=1,\ldots,N}\left|\alpha_j - \frac{\kappa_l^{(j)}}{\xi_l}\right| \leqslant \frac{C}{(\xi_l)^l}, \quad \forall l \in \mathbb{N}.$$

Let

$$J \doteq \left\{ j \in \{1,\ldots,n\},\ b_j \equiv 0 \right\} = \{j_1 < \cdots < j_m\}.$$

The main result of this work is the following theorem:

Theorem 2.2. *With the above notation, the operator $\mathbb{L}$ is globally solvable if and only if one of the following two situations occurs*:

I) *$J \neq \emptyset$ and $(a_{j_1 0}, \ldots, a_{j_m 0}) \notin \mathbb{Q}^m$ is non-Liouville.*
II) *Any primitive, in the minimal covering $\mathcal{T}$, of Π^*b has only connected sublevels and superlevels and, additionally, one of the following conditions holds*:

1. *$J = \emptyset$, b is exact and $a_0 \in \mathbb{Z}^n$*;
2. *$J \neq \emptyset$, b is exact, $a_0 \in \mathbb{Q}^n$ and $q_* = q_J$*;
3. *b is not exact.*

Note that when $J = \{1, \ldots, n\}$ we have that $b = 0$, hence b is exact, and $q_* = q_J$. In this case Theorem 2.2 says that $\mathbb{L}$ is globally solvable if and only if either $a_0 \notin \mathbb{Q}^n$ is non-Liouville or $a_0 \in \mathbb{Q}^n$, which was proved by Bergamasco and Petronilho in [7]. Thus, in order to prove Theorem 2.2 it suffices to prove the following propositions:

Proposition 2.3. *If $J = \emptyset$, then the operator $\mathbb{L}$ is globally solvable if and only if one of the following conditions holds*:

i) *There is a function b_j that does not change sign.*
ii) *b is exact, $a_0 \in \mathbb{Z}^n$ and the sublevels $\Omega_s = \{t \in \mathbb{T}^n,\ B(t) < s\}$ and superlevels $\Omega^s = \{t \in \mathbb{T}^n,\ B(t) > s\}$ are connected for every $s \in \mathbb{R}$.*

Proposition 2.4. *If $\emptyset \neq J \neq \{1, \ldots, n\}$ then the operator $\mathbb{L}$ is globally solvable if and only if one of the following conditions holds*:

i) *There is a function $b_j \not\equiv 0$ that does not change sign.*
ii) *$(a_{j_1 0}, \ldots, a_{j_m 0}) \notin \mathbb{Q}^m$ is non-Liouville.*
iii) *$a_0 \in \mathbb{Q}^n$, $q_* = q_J$, b is exact and the sublevels $\Omega_s = \{t \in \mathbb{T}^n,\ B(t) < s\}$ and superlevels $\Omega^s = \{t \in \mathbb{T}^n,\ B(t) > s\}$ are connected for every $s \in \mathbb{R}$.*

Proposition 2.4 provides the following interesting example.

Example 2.5. The system

$$\begin{cases} L_1 = \dfrac{\partial}{\partial t_1} + \dfrac{1}{4}\dfrac{\partial}{\partial x}, \\ L_2 = \dfrac{\partial}{\partial t_2} + \left(\dfrac{1}{2} + i \sin t_2\right)\dfrac{\partial}{\partial x} \end{cases}$$

is globally solvable on $\mathbb{T}^3$ since $q_J = q_* = 4$, whereas

$$\begin{cases} L_1 = \dfrac{\partial}{\partial t_1} + \dfrac{1}{2}\dfrac{\partial}{\partial x}, \\ L_2 = \dfrac{\partial}{\partial t_2} + \left(\dfrac{1}{4} + i \sin t_2\right)\dfrac{\partial}{\partial x} \end{cases}$$

is not globally solvable since in this case $q_J = 2 < 4 = q_*$.

When b is not exact, condition (i) in Propositions 2.3 and 2.4 is equivalent to the property of all sublevels and superlevels sets of a primitive B being connected in $\mathcal{T}$. Indeed, this result will be proved in the following lemma:

Lemma 2.6. *If $b = \sum_{j=1}^n b_j(t_j)\,dt_j$ is not exact and $\mathcal{T}$ denotes the minimal covering, then the sublevels $\Omega_s = \{t \in \mathcal{T},\ B(t) = \sum_{j=1}^n B_j(t_j) < s\}$ and superlevels $\Omega^s = \{t \in \mathcal{T},\ B(t) = \sum_{j=1}^n B_j(t_j) > s\}$ are all connected for every $s \in \mathbb{R}$ if and only if there exists a function $b_j \not\equiv 0$ that does not change sign.*

Proof. Suppose that each $b_j \not\equiv 0$ changes sign. Let, as before, r be the rank of the group of the periods of b. We may assume that $b_1, \dots, b_r$ do not have primitives in $\mathbb{T}$ and $b_{r+1}, \dots, b_n$ do. Then, for each $j = 1, \dots, r$, the function B_j has a disconnected superlevel on $\mathbb{R}$, hence there are a real number r_j and an open bounded interval I_j which is a connected component of $\{t_j \in \mathbb{R},\ B_j(t_j) > r_j\}$. We write $I_j = (p_j, q_j)$ and $M_j \doteq \max\{B_j(t_j),\ t_j \in [p_j, q_j]\}$, and we have $B_j(p_j) < M_j$ and $B_j(q_j) < M_j$. Also, we will denote by M' the maximum of $B_{r+1} + \cdots + B_n$ over $\mathbb{T}^{n-r}$.

Consider now the sets $I = I_1 \times \cdots \times I_r$ and $U \doteq I \times \mathbb{T}^{n-r}$. We take $t^* = (t_1^*, \dots, t_n^*) \in U$ where $B_j(t_j^*) = M_j$, $j = 1, \dots, r$ and $B_{r+1}(t_{r+1}^*) + \cdots + B_n(t_n^*) = M'$. Thus, if $M \doteq B(t^*)$ then $B(t) \leqslant M$, $\forall t \in U$ and $B(t) < M$, $\forall t \in \partial I \times \mathbb{T}^{n-r}$.

We choose $\widetilde{M} < s < M$ where $\widetilde{M} = \max\{B(t);\ t \in \partial I \times \mathbb{T}^{n-r}\}$; we claim that the superlevel Ω^s is disconnected in $\mathcal{T}$. Indeed, we write $\Omega^s = (U \cap \Omega^s) \cup (\Omega^s \setminus U)$ hence $U \cap \Omega^s \neq \emptyset$ and $\Omega^s \setminus U \neq \emptyset$. Since U is an open set in $\mathcal{T}$ hence $U \cap \Omega^s$ is an open set in Ω^s (in the topology induced by $\mathcal{T}$). It remains now to prove that $\Omega^s \setminus U$ is open in Ω^s.

Let $t = (t', t'') \in \Omega^s \setminus U = \Omega^s \cap (\mathcal{T} \setminus U)$ where $t' = (t_1, \dots, t_r)$ and $t'' = (t_{r+1}, \dots, t_n)$, thus $t' \in \mathbb{R}^r \setminus I$. Hence, there exists a neighborhood $V_{t'}$ of t' such that $V_{t'} \subset \mathbb{R}^r \setminus I$. Otherwise, if $V_{t'} \cap I \neq \emptyset$ for each neighborhood $V_{t'} \subset \mathbb{R}^r$ of t' then $t' \in \partial I$, thus $B(t) \leqslant \widetilde{M} < s$ that is a contradiction. We define $V_t = V_{t'} \times \mathbb{T}^{n-r}$, then $V_t \subset \mathcal{T} \setminus U$. Also, there exists a neighborhood $W_t \subset \mathcal{T}$ of t such that $W_t \subset \Omega^s$. Therefore $(V_t \cap W_t) \subset \Omega^s \cap (\mathcal{T} \setminus U)$ and we conclude that $\Omega^s \setminus U$ is an open set in Ω^s.

On the other hand, suppose that there is $b_j \not\equiv 0$ that does not change sign. We may assume that $b_j(t_j) \leqslant 0$ $\forall t_j \in \mathbb{R}$. Thus B_j is a non-increasing function on $\mathbb{R}$.

Let $s \in \mathbb{R}$ and $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n) \in \Omega_s$. We suppose that $p_j \leqslant q_j$. For each $k = 1, \dots, n$, $k \neq j$ there is a path σ_k (in $\mathbb{R}$ or $\mathbb{S}^1$) connecting q_k and p_k. Let M be the largest $M_k = \max\{B_k(\tau),\ \tau \in \sigma_k\}$ and $C > 0$ such that $(n-1)M + B_j(q_j + C) < s$.

Since B is of the form $B(t) = \sum_{k=1}^n B_k(t_k)$ and B_j is a non-increasing function there exists a path γ_1 connecting q and $(q_1, \dots, q_j + C, \dots, q_n)$ such that $\gamma_1 \subset \Omega_s$. Let γ_2 be the path $\gamma_2 \doteq (\sigma_1, \dots, q_j + C, \dots, \sigma_n)$, therefore $B(\gamma_2(\xi)) \leqslant (n-1)M + B_j(q_j + C) < s$, $\xi \in \mathbb{R}$. Hence, $\gamma_2 \subset \Omega_s$ is a path connecting $(q_1, \dots, q_j + C, \dots, q_n)$ and $(p_1, \dots, q_j + C, \dots, p_n)$.

Finally, since $p \in \Omega_s$ and $p_j \leqslant q_j + C$ there exists a path $\gamma_3 \subset \Omega_s$ connecting p and $(p_1, \dots, q_j + C, \dots, p_n)$. We conclude that the sublevel Ω_s is path-connected. Similarly, the superlevel Ω^s is path-connected. $\square$

The differential operator $\mathbb{L}$ is globally solvable if and only if the differential operator

$$d_t + \big(a_0 + ib(t)\big) \wedge \frac{\partial}{\partial x} \tag{2.2}$$

is globally solvable. Indeed, we write a in the form $a = a_0 + d_t A$ where $A \in C^\infty(\mathbb{T}^n; \mathbb{R})$ and define

$$\begin{aligned} S : \mathcal{D}'\big(\mathbb{T}^{n+1}\big) &\longrightarrow \mathcal{D}'\big(\mathbb{T}^{n+1}\big), \\ \sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi) e^{i\xi x} &\longmapsto \sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi) e^{i\xi A(t)} e^{i\xi x}. \end{aligned}$$

Observe that S is an automorphism of $\mathcal{D}'(\mathbb{T}^{n+1})$ and of $C^\infty(\mathbb{T}^{n+1})$. Also, the following equality holds:

$$S\mathbb{L}S^{-1} = d_t + \big(a_0 + ib(t)\big) \wedge \frac{\partial}{\partial x},$$

which ensures our statement. Therefore, it is enough to prove Propositions 2.3 and 2.4 for the operator (2.2). From now on we will denote by $\mathbb{L}$ the operator (2.2), that is,

$$\mathbb{L} = d_t + \big(a_0 + ib(t)\big) \wedge \frac{\partial}{\partial x} \tag{2.3}$$

and by $\mathbb{E}$ the corresponding space of compatibility conditions. The new operator $\mathbb{L}$ is associated with the vector fields

$$L_j = \frac{\partial}{\partial t_j} + \big(a_{j0} + ib_j(t_j)\big)\frac{\partial}{\partial x}, \quad j = 1, \dots, n. \tag{2.4}$$

3. Sufficiency in Proposition 2.3

Suppose first that there exists a function $b_j \not\equiv 0$ that does not change sign. We may assume that $b_j(t_j) \leqslant 0$, otherwise it suffices to use the diffeomorphism $(t_1, \dots, t_j, \dots, t_n, x) \mapsto (t_1, \dots, -t_j, \dots, t_n, x)$ to obtain new coordinates in $\mathbb{T}^{n+1}$ where this property holds. Therefore, the primitive B_j is a non-increasing function and $b_{j0} < 0$.

Let $\mathbb{L}$ defined by (2.3) and $f \in \mathbb{E}$. In order to construct a global solution to the equation $\mathbb{L}u = f$ we will use x-Fourier coefficients and the formal series

$$u(t, x) = \sum_{\xi \in \mathbb{Z}} \hat{u}(t, \xi) e^{i\xi x} \tag{3.1}$$

and

$$f_k(t, x) = \sum_{\xi \in \mathbb{Z}} \hat{f}_k(t, \xi) e^{i\xi x}, \quad k = 1, \dots, n. \tag{3.2}$$

Substituting the series (3.1) and (3.2) in the equation $L_j u = f_j$, with L_j given by (2.4), we obtain for each $\xi \in \mathbb{Z}$ the following equation

$$\frac{\partial}{\partial t_j}\hat{u}(t, \xi) + i\xi\big(a_{j0} + ib_j(t_j)\big)\hat{u}(t, \xi) = \hat{f}_j(t, \xi).$$

Since $b_{j0} \neq 0$, the preceding equation, for each $\xi \neq 0$, has exactly one solution given by

$$\begin{aligned}\hat{u}(t, \xi) = {} & d_{j\xi} \int_0^{t_j} e^{i\xi(\mathcal{C}_j(s_j) - \mathcal{C}_j(t_j))} \hat{f}_j(t_1, \dots, s_j, \dots, t_n, \xi)\, ds_j \\ & + (d_{j\xi} - 1) \int_{t_j}^{2\pi} e^{i\xi(\mathcal{C}_j(s_j) - \mathcal{C}_j(t_j))} \hat{f}_j(t_1, \dots, s_j, \dots, t_n, \xi)\, ds_j,\end{aligned} \tag{3.3}$$

where

$$d_{j\xi} = \big(1 - e^{-i\xi 2\pi c_{j0}}\big)^{-1} \quad \text{and} \quad \mathcal{C}_j(t_j) = a_{j0}t_j + iB_j(t_j).$$

When $\xi = 0$ we must solve

$$\frac{\partial}{\partial t_j}\hat{u}(t, 0) = \hat{f}_j(t, 0);$$

a solution is

$$\hat{u}(t,0)=\int_0^{t_j}\hat{f}_j(t_1,\dots,s_j,\dots,t_n,0)\,ds_j \tag{3.4}$$

which is well defined and is a solution because $\hat{f}(t,0)$ is exact (see (2.1)).

We will prove that u given by (3.1) with x-Fourier coefficients (3.3) and (3.4) is a global solution of the equation $\mathbb{L}u=f$.

Indeed, since $b_{j0}<0$ there exists a constant $C>0$ such that

$$C^{-1}<|d_{j\xi}|<C, \quad \forall\xi\in\mathbb{Z}_+ \quad \text{and} \quad C^{-1}<|d_{j\xi}-1|<C, \quad \forall\xi\in\mathbb{Z}_-.$$

For each $\xi\in\mathbb{Z}_+$ we write the formula (3.3) as follows

$$\hat{u}(t,\xi)=d_{j\xi}\int_0^{t_j}e^{i\xi(\mathcal{C}_j(s_j)-\mathcal{C}_j(t_j))}\hat{f}_j(t_1,\dots,s_j,\dots,t_n,\xi)\,ds_j$$
$$+\underbrace{e^{i\xi 2\pi c_{j0}}(d_{j\xi}-1)}_{d_{j\xi}}\int_{t_j}^{2\pi}e^{i\xi(\mathcal{C}_j(s_j)-\mathcal{C}_j(t_j)-2\pi c_{j0})}\hat{f}_j(t_1,\dots,s_j,\dots,t_n,\xi)\,ds_j.$$

Note that in the first integral we have $0\leqslant s_j\leqslant t_j$, hence $B_j(t_j)-B_j(s_j)\leqslant 0$, while in the second integral $t_j\leqslant s_j\leqslant 2\pi$ and $B_j(t_j)-B_j(s_j)+2\pi b_{j0}\leqslant 0$. Thus, from the preceding equation we obtain

$$\left|\hat{u}(t,\xi)\right|\leqslant 2\pi C\max_{s\in\mathbb{T}^n}\left|\hat{f}_j(s,\xi)\right|, \quad \forall t\in\mathbb{T}^n,\ \forall\xi\in\mathbb{Z}_+.$$

Similarly, for each $\xi\in\mathbb{Z}_-$ we write the formula (3.3) in the form

$$\hat{u}(t,\xi)=\underbrace{e^{-i\xi 2\pi c_{j0}}d_{j\xi}}_{d_{j\xi}-1}\int_0^{t_j}e^{i\xi(\mathcal{C}_j(s_j)-\mathcal{C}_j(t_j)+2\pi c_{j0})}\hat{f}_j(t_1,\dots,s_j,\dots,t_n,\xi)\,ds_j$$
$$+(d_{j\xi}-1)\int_{t_j}^{2\pi}e^{i\xi(\mathcal{C}_j(s_j)-\mathcal{C}_j(t_j))}\hat{f}_j(t_1,\dots,s_j,\dots,t_n,\xi)\,ds_j.$$

Now, in the second integral we have $t_j\leqslant s_j\leqslant 2\pi$ and $B_j(t_j)-B_j(s_j)\geqslant 0$, whereas in the first integral we have $0\leqslant s_j\leqslant t_j$ and $B_j(t_j)-B_j(s_j)-2\pi b_{j0}\geqslant 0$. Again, we obtain

$$\left|\hat{u}(t,\xi)\right|\leqslant 2\pi C\max_{s\in\mathbb{T}^n}\left|\hat{f}_j(s,\xi)\right|, \quad \forall t\in\mathbb{T}^n,\ \forall\xi\in\mathbb{Z}_-.$$

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let $\beta = \alpha - \alpha_j e_j$, where $e_j = (0, \ldots, 1, \ldots, 0)$. We have

$$\partial^\beta \hat{u}(t,\xi) = d_{j\xi} \int_0^{t_j} e^{i\xi(\mathcal{C}_j(s_j) - \mathcal{C}_j(t_j))} \partial^\beta \hat{f}_j(t_1, \ldots, s_j, \ldots, t_n, \xi)\, ds_j$$
$$+ (d_{j\xi} - 1) \int_{t_j}^{2\pi} e^{i\xi(\mathcal{C}_j(s_j) - \mathcal{C}_j(t_j))} \partial^\beta \hat{f}_j(t_1, \ldots, s_j, \ldots, t_n, \xi)\, ds_j.$$

Note that, as before we have the following inequality

$$\left|\partial^\beta \hat{u}(t,\xi)\right| \leqslant 2\pi C \max_{s \in \mathbb{T}^n} \left|\partial^\beta \hat{f}_j(s,\xi)\right|, \quad \forall t \in \mathbb{T}^n,\ \forall \xi \in \mathbb{Z}.$$

For $m \in \mathbb{N}$ we have

$$\frac{\partial^m}{\partial t_j^m} \partial^\beta \hat{u}(t,\xi) = P_\xi(t_j) \partial^\beta \hat{u}(t,\xi) + Q_\xi(t), \tag{3.5}$$

where P_ξ is a polynomial involving only powers of ξ (powers less than m) and derivatives of c_j which are bounded on $\mathbb{T}^n$, and Q_ξ is a polynomial involving powers of ξ (powers less than $m-1$) and derivatives of c_j and $\partial^\beta \hat{f}_j(t,\xi)$. Thus, there exists a constant $C_m > 0$ such that

$$\left|P_\xi(t_j)\right| \leqslant C_m |\xi|^m \quad \text{and} \quad \left|Q_\xi(t)\right| \leqslant C_m |\xi|^m \sum_{k=0}^{m-1} \left| \frac{\partial^k}{\partial t_j^k} \partial^\beta \hat{f}_j(t,\xi)\right|, \quad \forall t \in \mathbb{T}^n.$$

If $m = \alpha_j$ then, from (3.5), it follows

$$\left|\partial^\alpha \hat{u}(t,\xi)\right| \leqslant C_{\alpha_j} |\xi|^{\alpha_j} \left(\left|\partial^\beta \hat{u}(t,\xi)\right| + \sum_{k=0}^{\alpha_j - 1} \left| \frac{\partial^k}{\partial t_j^k} \partial^\beta \hat{f}_j(t,\xi)\right| \right)$$
$$\leqslant \widetilde{C}_{\alpha_j} |\xi|^{\alpha_j} \sum_{k=0}^{\alpha_j - 1} \max_{s \in \mathbb{T}^n} \left| \frac{\partial^k}{\partial t_j^k} \partial^\beta \hat{f}_j(s,\xi)\right|, \quad \forall t \in \mathbb{T}^n,\ \forall \xi \in \mathbb{Z},$$

and if $\alpha_j = 0$ we have

$$\left|\partial^\alpha \hat{u}(t,\xi)\right| \leqslant 2\pi C \max_{s \in \mathbb{T}^n} \left|\partial^\alpha \hat{f}_j(s,\xi)\right|, \quad \forall t \in \mathbb{T}^n,\ \forall \xi \in \mathbb{Z}.$$

Since f_j is a smooth function on $\mathbb{T}^{n+1}$ we conclude that

$$u(t,x) = \sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi) e^{i\xi x} \in C^\infty\left(\mathbb{T}^{n+1}\right).$$

It is not difficult to check that u is indeed a solution of $L_k u = f_k$ for $k = 1, \ldots, n$. This finishes the proof of sufficiency in the case when i) holds.

Suppose now that b is exact, $a_0 \in \mathbb{Z}^n$ and the sets $\Omega_s = \{t \in \mathbb{T}^n,\ B(t) < s\}$ and $\Omega^s = \{t \in \mathbb{T}^n,\ B(t) > s\}$ are connected for every $s \in \mathbb{R}$, where B is a global primitive of b.

Since $a_0 \in \mathbb{Z}^n$, we define

$$\begin{aligned} S : \mathcal{D}'\big(\mathbb{T}^{n+1}\big) &\longrightarrow \mathcal{D}'\big(\mathbb{T}^{n+1}\big), \\ \sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi) e^{i\xi x} &\longmapsto \sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi) e^{-i\xi a_0 \cdot t} e^{i\xi x}. \end{aligned}$$

Again, S defines an automorphism of $\mathcal{D}'(\mathbb{T}^{n+1})$ and of $C^\infty(\mathbb{T}^{n+1})$. Also, the following relation holds:

$$S^{-1} \mathbb{L} S = \mathbb{L}_0 \doteq d_t + ib(t) \wedge \frac{\partial}{\partial x}. \tag{3.6}$$

Since b is exact, the work [9] implies that the operator $\mathbb{L}_0$ is globally solvable if and only if any global primitive of b has only connected sublevels and superlevels on $\mathbb{T}^n$. Therefore, $\mathbb{L}_0$ is globally solvable and by (3.6) we conclude that $\mathbb{L}$ is globally solvable.

4. Necessity in Proposition 2.3

We are going to prove that if both conditions i) and ii) fail then $\mathbb{L}$ is not globally solvable.

Let $B : \mathbb{R}^n \to \mathbb{R}$ be a global primitive of the pull-back $\Pi^* b$ via the universal covering $\Pi : \mathbb{R}^n \to \mathbb{T}^n$. Since each b_j changes sign and B is of the form $B(t) = \sum_{j=1}^n B_j(t_j)$ we may assume the following property:

⋆ The maximum of B over $[0, 2\pi]^n$ is not attained at the boundary.

Furthermore, we may – and will – assume that $B(0) = 0$ and for each j the period $b_{j0} \leqslant 0$. Otherwise if, say, $b_{j0} > 0$ it suffices to use the diffeomorphism

$$(t_1, \dots, t_j, \dots, t_n) \longmapsto (t_1, \dots, -t_j, \dots, t_n)$$

to obtain new coordinates on the torus $\mathbb{T}_t^n$ such that the new $b_{j0} < 0$.

Suppose first that b is exact, $a_0 \in \mathbb{Z}^n$ and the global primitive of b has a disconnected sublevel or superlevel on $\mathbb{T}^n$. Then, the operator $\mathbb{L}_0 = d_t + ib(t) \wedge \frac{\partial}{\partial x}$ is not globally solvable by [9], hence $\mathbb{L}$ is not globally solvable by (3.6).

Suppose now that either b is not exact or $a_0 \notin \mathbb{Z}^n$, which is equivalent to $c_0 = (c_{10}, \dots, c_{n0}) \notin \mathbb{Z}^n$. We consider two cases:

Case 1: $\{c_{10}, \dots, c_{n0}\} \cap \mathbb{Z} = \emptyset$;
Case 2: $\{c_{10}, \dots, c_{n0}\} \cap \mathbb{Z} \neq \emptyset$.

We now move on to the proof of nonsolvability of $\mathbb{L}$ in Case 1. This proof will be given by Propositions 4.1–4.3.

Case 1: $\{c_{10}, \dots, c_{n0}\} \cap \mathbb{Z} = \emptyset$.

In order to construct an $f \in \mathbb{E}$ such that the equation $\mathbb{L}u = f$ has no solution $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ we need to define an f satisfying the compatibility condition $\mathbb{L}f = 0$ or equivalently $L_j f_k - L_k f_j = 0$, $k, j = 1, \dots, n$. This condition together with the property $[L_j, L_k] = 0$, $k, j = 1, \dots, n$ imply the following equality:

$$L_n L_{n-1} \dots L_1 u = L_{\sigma(n)} L_{\sigma(n-1)} \dots L_{\sigma(1)} u, \quad \sigma \in \mathcal{S}_n, \tag{4.1}$$

where $\mathcal{S}_n$ is the group of all permutations of the natural numbers $1, \ldots, n$. The plan is to use (4.1) to find the x-Fourier coefficients of the functions f_j.

Consider the set $\mathcal{F} = \{c_{10}, \ldots, c_{n0}\} \cap \mathbb{Q}$. If $c_{j0} \in \mathcal{F}$, we write $c_{j0} = r_j/s_j$, where $r_j \in \mathbb{Z}$ and $s_j \in \mathbb{N}$ are coprime, and we define the set

$$\mathcal{G} = \bigcup_{c_{j0} \in \mathcal{F}} s_j \mathbb{Z}. \tag{4.2}$$

Since each $c_{j0} \notin \mathbb{Z}$ we have that $s_j > 1$ and that both $\mathbb{Z}_+ \setminus \mathcal{G}$ and $\mathbb{Z}_- \setminus \mathcal{G}$ have infinitely many elements.

Motivated by (4.1), we substitute the formal series

$$u(t,x) = \sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi) e^{i\xi x} \quad \text{and} \quad h(t,x) = \sum_{\xi \in \mathbb{Z}} \hat{h}(t,\xi) e^{i\xi x}$$

in the equation

$$L_n L_{n-1} \ldots L_1 u \doteq h,$$

where the function h will be chosen later on. We obtain

$$\hat{h}(t,\xi) = \big(\partial/\partial t_n + i\xi c_n(t_n)\big) \cdots \big(\partial/\partial t_1 + i\xi c_1(t_1)\big) \hat{u}(t,\xi), \quad \xi \in \mathbb{Z},$$

where $c_j(t_j) = a_{j0} + i b_j(t_j)$. For each $\xi \in \mathbb{Z} \setminus \mathcal{G}$ the preceding equation has exactly one solution given by

$$\hat{u}(t,\xi) = d_\xi \int_{[0,2\pi]^n} e^{i\xi(\mathcal{C}(t-s) - \mathcal{C}(t))} \hat{h}(t-s,\xi)\, ds, \tag{4.3}$$

where $d_\xi = \prod_{j=1}^n d_{j\xi}$ with $d_{j\xi} = (1 - e^{-i2\pi\xi c_{j0}})^{-1}$ and $\mathcal{C}(t) = a_0 \cdot t + iB(t)$.

Let $v(t,\xi)$ be given by the right-hand side of (4.3). For each $\xi \in \mathbb{Z} \setminus \mathcal{G}$ we define

$$\hat{f}_j(t,\xi) \doteq \big(\partial_{t_j} + i\xi c_j(t_j)\big) v(t,\xi), \quad j = 1, \ldots, n,$$

hence

$$\hat{f}_j(t,\xi) = \frac{d_\xi}{d_{j\xi}} \int_{[0,2\pi]^{n-1}} e^{i\xi(\mathcal{C}(t-s+s_j e_j) - \mathcal{C}(t))} \hat{h}(t - s + s_j e_j, \xi)\, ds^{(j)}, \tag{4.4}$$

where $ds^{(j)} = ds_1 \ldots ds_{j-1}\, ds_{j+1} \ldots ds_n$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

We define

$$f_j(t,x) \doteq \sum_{\xi \in \mathbb{Z} \setminus \mathcal{G}} \hat{f}_j(t,\xi) e^{i\xi x}, \quad j = 1, \ldots, n.$$

In Proposition 4.2 we prove that $f_1, \ldots, f_n$ are smooth functions for a convenient choice of function h. It will then follow from the definition of $\hat{f}_j$ that $f(t,x) = \sum_{j=1}^n f_j(t,x)\, dt_j \in \mathbb{E}$. Also, if there existed u such that $\mathbb{L}u = f$ then the coefficients would have to verify $\hat{u}(t,\xi) = v(t,\xi)$.

Construction of the function h: Let $\chi_\delta \in C_c^\infty(\mathbb{R}^n)$ be a function such that

$$\chi_\delta(t) = \begin{cases} 0 & \text{if } t \in \mathbb{R}^n \backslash (-2\delta, 2\delta)^n, \\ 1 & \text{if } t \in (-\delta, \delta)^n \end{cases}$$

and $0 \leqslant \chi_\delta(t) \leqslant 1$, for all $t \in \mathbb{R}^n$, and some $0 < \delta < \pi/2$.

We will denote by M the maximum of B over the cube $[0, 2\pi]^n$ and by M_j the maximum of B over the set $[0, 2\pi]^n \cap \{t_j = 0\}$, $j = 1, \dots, n$. We define the coefficients

$$\hat{h}(t, \xi) = \begin{cases} (d_\xi)^{-1} e^{-\xi(M + Kq(t) - \lambda - ip(t))} & \text{if } \xi \in \mathbb{Z}_+ \backslash \mathcal{G}, \\ 0 & \text{if } \xi \in \mathbb{Z}_- \cup \mathcal{G}, \end{cases} \tag{4.5}$$

where $0 < \lambda < \frac{M - \widetilde{M}}{2}$ ($\widetilde{M}$ is the largest M_j) and $K > 0$ are constants. The constant K will be chosen later on. Furthermore, p and q are smooth periodic functions as follows

$$q(t) = 1 + \sum_{\nu \in \mathbb{Z}^n} \big(|t + 2\pi\nu|^2 - 1 \big) \chi_\delta(t + 2\pi\nu), \quad t \in \mathbb{R}^n$$

and

$$p(t) = \sum_{\nu \in \mathbb{Z}^n} a_0 \cdot \big(t^* - t - 2\pi\nu \big) \chi_\delta(t + 2\pi\nu), \quad t \in \mathbb{R}^n,$$

where $t^* \in (0, 2\pi)^n$ is such that $B(t^*) = M$. Note that

- $q(t) \geqslant 0$ $\forall t \in \mathbb{R}^n$ and $q(t) = 0$ if and only if $t = 2\pi\nu$, $\nu \in \mathbb{Z}^n$.
- If $|t| < \delta$ then $p(t) = a_0 \cdot (t^* - t)$ and $q(t) = |t|^2$.

Finally, we define

$$h(t, x) = \sum_{\xi \in \mathbb{Z}_+ \backslash \mathcal{G}} \hat{h}(t, \xi) e^{i\xi x}.$$

Proposition 4.1. $h \in C^\infty(\mathbb{T}^{n+1})$.

Proof. Since $0 < \lambda < M$, we have $(M + Kq(t) - \lambda) \geqslant M - \lambda > 0$. Also, there exists a constant $C > 0$ such that $|(d_\xi)^{-1}| \leqslant C$ for all $\xi \in \mathbb{Z}_+ \backslash \mathcal{G}$, thus

$$\big| \hat{h}(t, \xi) \big| \leqslant C e^{-\xi(M - \lambda)}, \quad \forall t \in [0, 2\pi]^n \text{ and } \forall \xi \in \mathbb{Z}_+ \backslash \mathcal{G}.$$

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we have $\partial^\alpha \hat{h}(t, \xi) = P_\xi(t) \hat{h}(t, \xi)$ where $P_\xi(t)$ is a polynomial involving only powers of ξ (powers less than $|\alpha| = \alpha_1 + \dots + \alpha_n$) and derivatives of p and q, which are bounded on $[0, 2\pi]^n$. Thus, there exists a constant $C_\alpha > 0$ such that $|P_\xi(t)| \leqslant C_\alpha \xi^{|\alpha|}$ for all $\xi \in \mathbb{Z}_+ \setminus \mathcal{G}$. Therefore,

$$\big| \partial^\alpha \hat{h}(t, \xi) \big| = \big| P_\xi(t) \big| \big| \hat{h}(t, \xi) \big| \leqslant C C_\alpha \xi^{|\alpha|} e^{-\xi(M - \lambda)}, \quad \xi \in \mathbb{Z}_+ \backslash \mathcal{G},$$

whence we conclude that $h \in C^\infty(\mathbb{T}^{n+1})$. $\square$

Substituting the coefficients (4.5) in Eqs. (4.3) and (4.4), for each $\xi \in \mathbb{Z}_+ \setminus \mathcal{G}$ we have

$$\hat{u}(t,\xi) = \int\limits_{[0,2\pi]^n} e^{\xi\varphi_1(t,s)} e^{i\xi\varphi_2(t,s)}\, ds, \tag{4.6}$$

and

$$\hat{f}_j(t,\xi) = \frac{1}{d_{j\xi}} \int\limits_{[0,2\pi]^{n-1}} e^{\xi\varphi_1(t,s-s_je_j)} e^{i\xi\varphi_2(t,s-s_je_j)}\, ds^{(j)}, \quad j = 1,\dots,n, \tag{4.7}$$

where

$$\varphi_1(t,s) \doteq B(t) - B(t-s) - Kq(t-s) - M + \lambda$$

and

$$\varphi_2(t,s) \doteq p(t-s) - a_0 \cdot s$$

are real valued functions.

Proposition 4.2. *$f_j \in C^\infty(\mathbb{T}^{n+1})$ for all $j = 1,\dots,n$.*

Proof. We identify $s - s_je_j$ with $(s_1,\dots,s_{j-1},s_{j+1},\dots,s_n)$. The set

$$\mathcal{Q}_j = \{(t,s-s_je_j) \in [0,2\pi]^{2n-1};\ q(t-s+s_je_j) = 0\}$$

is closed since the function $[0,2\pi]^{2n-1} \ni (t,s-s_je_j) \mapsto q(t-s+s_je_j)$ is continuous. Since $q(t-s+s_je_j) = 0$ if and only if $t - s + s_je_j = 2\nu\pi$, $\nu \in \mathbb{Z}^n$, we obtain

$$\begin{aligned}
\varphi_1(t,s-s_je_j) &= B(t) - B(t-s+s_je_j) - Kq(t-s+s_je_j) - M + \lambda \\
&= B(s-s_je_j+2\nu\pi) - B(2\nu\pi) - M + \lambda \\
&= B(s-s_je_j) - M + \lambda \\
&\leqslant \widetilde{M} - M + \lambda < \frac{\widetilde{M}-M}{2}, \quad (t,s-s_je_j) \in \mathcal{Q}_j.
\end{aligned}$$

Take an open set U such that $U \supset \mathcal{Q}_j$ and

$$\varphi_1(t,s-s_je_j) \leqslant \frac{\widetilde{M}-M}{2}, \quad (t,s-s_je_j) \in U.$$

If $\sigma \doteq \min\{q(t-s+s_je_j);\ (t,s-s_je_j) \in [0,2\pi]^{2n-1}\backslash U\}$ then $\sigma > 0$. Therefore, if necessary, we take a larger $K > 0$ to obtain

$$\begin{aligned}
\varphi_1(t,s-s_je_j) &\leqslant -\mu - K\sigma + \lambda \\
&\leqslant \frac{\widetilde{M}-M}{2}, \quad (t,s-s_je_j) \in [0,2\pi]^{2n-1}\backslash U.
\end{aligned}$$

where $\mu = \min\{B(\tau), \tau \in [0,2\pi]^{2n-1}\backslash U\}$.

Since $\varphi_1(t, s - s_j e_j) \leqslant \frac{\widetilde{M}-M}{2}$ for all $(t, s - s_j e_j) \in [0, 2\pi]^{2n-1}$ we obtain from (4.7)

$$\left|\hat{f}_j(t,\xi)\right| \leqslant Ce^{-\xi\frac{M-\widetilde{M}}{2}} \quad \forall\xi \in \mathbb{Z}_+ \setminus \mathcal{G},$$

where the constant $C > 0$ does not depend on t and ξ.

Let $\varphi(t,s) = \varphi_1(t,s) + i\varphi_2(t,s)$. Given $\alpha \in \mathbb{N}^n$, it follows from (4.7) that

$$\partial^\alpha \hat{f}_j(t,\xi) = \frac{1}{d_{j\xi}} \int\limits_{[0,2\pi]^{n-1}} P_\xi(t,s) e^{\xi\varphi(t,s-s_j e_j)}\, ds^{(j)},$$

where $P_\xi(t,s)$ is a polynomial involving only powers (at most $|\alpha|$) of ξ and derivatives of φ, which are bounded on $[0, 2\pi]^{2n}$. Thus, there exists a constant $C_\alpha > 0$ such that $|P_\xi(t)| \leqslant C_\alpha \xi^{|\alpha|}$. Therefore,

$$\begin{aligned}
\left|\partial^\alpha \hat{f}_j(t,\xi)\right| &\leqslant \frac{1}{|d_{j\xi}|} \int\limits_{[0,2\pi]^{n-1}} \left|P_\xi(t,s)\right| \left|e^{\xi\varphi(t,s-s_j e_j)}\right| ds^{(j)} \\
&\leqslant C_\alpha C \xi^{|\alpha|} e^{-\xi\frac{M-\widetilde{M}}{2}}.
\end{aligned}$$

We conclude that $f_j \in C^\infty(\mathbb{T}^{n+1})$. $\square$

Proposition 4.3. *There is no distribution $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ whose sequence of x-Fourier coefficients is the sequence* (4.6).

Proof. Since $B(0) = 0$, let $0 < \delta_0 < \delta$ such that $|B(t)| < \lambda/2$ for all $|t| < \delta_0$. Therefore, we consider the equation

$$\hat{u}\big(t^*, \xi\big) = I_\xi + J_\xi,$$

where $t^* \in (0, 2\pi)^n$ such that $B(t^*) = M$ and

$$\begin{aligned}
I_\xi &= \int\limits_{|s-t^*|<\delta_0} e^{i\xi(p(t^*-s)-a_0.s)} e^{-\xi(B(t^*-s)+Kq(t^*-s)-\lambda)}\, ds, \\
J_\xi &= \int\limits_{|s-t^*|\geqslant\delta_0} e^{i\xi(p(t^*-s)-a_0.s)} e^{-\xi(B(t^*-s)+Kq(t^*-s)-\lambda)}\, ds.
\end{aligned}$$

If we make the change of variables $\sigma = \sqrt{K\xi}(t^* - s)$ we obtain

$$\begin{aligned}
I_\xi &= \int\limits_{|s-t^*|<\delta_0} e^{-\xi(B(t^*-s)+K|t^*-s|^2-\lambda)}\, ds \\
&\geqslant e^{\xi\lambda/2} \int\limits_{|s-t^*|<\delta_0} e^{-\xi K|t^*-s|^2}\, ds \\
&= e^{\xi\lambda/2} \frac{1}{(K\xi)^{n/2}} \int\limits_{|\sigma|<\delta_0\sqrt{K\xi}} e^{-|\sigma|^2}\, d\sigma
\end{aligned}$$

$$= e^{\xi\lambda/2}\frac{1}{(K\xi)^{n/2}}\mu_\xi$$
$$\geqslant \frac{\mu_1 e^{\xi\lambda/2}}{(K\xi)^{n/2}},$$

where $\mu_\xi = \int_{|\sigma|<\delta_0\sqrt{K\xi}} e^{-|\sigma|^2}\,d\sigma$, $\xi \in \mathbb{Z}_+\backslash\mathcal{G}$, is an increasing sequence of real numbers.

On the other hand, we consider the cube $Q^* = t^* - [0,2\pi]^n$ and $\sigma \doteq \min_{\tau\in Q^*,|\tau|\geqslant\delta_0} q(\tau)$, thus $\sigma > 0$. Now, taking $K > \sigma^{-1}(3\lambda/2 - \min_{\tau\in Q^*,|\tau|\geqslant\delta_0} B(\tau))$ we have

$$|J_\xi| \leqslant \int\limits_{|s-t^*|\geqslant\delta_0} \left|e^{i\xi(p(t^*-s)-a_0.s)}\right| e^{-\xi(B(t^*-s)+Kq(t^*-s)-\lambda)}\,ds$$
$$\leqslant \int\limits_{|s-t^*|\geqslant\delta_0} e^{-\xi(B(t^*-s)+K\sigma-\lambda)}\,ds$$
$$\leqslant \int\limits_{|s-t^*|\geqslant\delta_0} e^{-\xi\lambda/2}\,ds = Ce^{-\xi\lambda/2},$$

where $C > 0$ is a constant that does not depend on ξ. Furthermore, there exists $\xi_0 \in \mathbb{Z}_+$ such that

$$\frac{\mu_1 e^{\xi\lambda/2}}{C(K\xi)^{n/2}} \geqslant 1, \quad \forall\xi \geqslant \xi_0.$$

Thus,

$$\left|\hat{u}\left(t^*,\xi\right)\right| \geqslant |I_\xi| - |J_\xi| \geqslant \frac{\mu_1 e^{\xi\lambda/2}}{(K\xi)^{n/2}} - Ce^{-\xi\lambda/2} \geqslant \frac{\mu_1}{(K\xi)^{n/2}}\left(e^{\xi\lambda/2} - 1\right)$$

for all $\xi \geqslant \xi_0$ with $\xi \in \mathbb{Z}_+\backslash\mathcal{G}$. Therefore, the sequence $\hat{u}(t^*,\xi)$ does not belong to any distribution $u \in \mathcal{D}'(\mathbb{T}^{n+1})$, which concludes the proof in Case 1.

We now move on to the proof in Case 2.

Case 2: $\{c_{10},\dots,c_{n0}\} \cap \mathbb{Z} \neq \emptyset.$

Recall that we are working under the assumption that $c_0 \notin \mathbb{Z}^n$. In this case, the proof consists in using *Case 1* together with the following lemma:

Lemma 4.4. *Let $1 \leqslant p \leqslant n$. If the involutive system generated by the vector fields $L_1,\dots,L_p$ is not globally solvable on $\mathbb{T}^{p+1}$ and if the periods $c_{(p+1)0},\dots,c_{n0} \in \mathbb{Z}$, then $\mathbb{L}$ is not globally solvable on $\mathbb{T}^{n+1}$.*

Proof. We denote by (t',t'',x) the coordinates of $\mathbb{T}^{n+1}$ where $t' = (t_1,\dots,t_p)$ and $t'' = (t_{p+1},\dots,t_n)$. Similarly, we use the notation $c_0 = (c_0', c_0'')$ where $c_0' = (c_{10},\dots,c_{p0})$ and $c_0'' = (c_{(p+1)0},\dots,c_{n0})$.

Consider the involutive system $\mathbb{L}^\#$ defined by the vector fields

$$L_j = \frac{\partial}{\partial t_j} + \left(a_{j0} + ib_j(t_j)\right)\frac{\partial}{\partial x}, \quad j = 1,\dots,p, \tag{4.8}$$

and its space of compatibility conditions $\mathbb{E}^\#$. Since (4.8) is not globally solvable on $\mathbb{T}^{p+1}_{(t',x)}$ there exists $g(t',x) = \sum_{j=1}^p g_j(t',x)\,dt_j \in \mathbb{E}^\#$ such that $\mathbb{L}^\# v = g$ has no solution $v \in \mathcal{D}'(\mathbb{T}^{p+1}_{(t',x)})$. Furthermore, since

$c_0'' \in \mathbb{Z}^{n-p}$ (hence $a_0'' \in \mathbb{Z}^{n-p}$ and $b_0'' = 0$) then $\tilde{b}(t'') \doteq b_{p+1}(t_{p+1})\,dt_{p+1} + \cdots + b_n(t_n)\,dt_n$ has a global primitive $\widetilde{B}(t'') \doteq B_{p+1}(t_{p+1}) + \cdots + B_n(t_n)$ on $\mathbb{T}^{n-p}_{t''}$.

We define

$$f_j(t,x) \doteq \sum_{\xi\in\mathbb{Z}} \hat{f}_j(t,\xi)e^{i\xi x}, \quad j = 1,\dots,p,$$

where

$$\hat{f}_j(t,\xi) \doteq \begin{cases} \hat{g}_j(t',\xi)e^{\xi(\widetilde{B}(t'')-M-ia_0''\cdot t'')} & \text{if } \xi \geqslant 0, \\ \hat{g}_j(t',\xi)e^{\xi(\widetilde{B}(t'')-m-ia_0''\cdot t'')} & \text{if } \xi < 0, \end{cases}$$

and $M = \max_{t''\in\mathbb{T}^{n-p}} \widetilde{B}(t'')$, $m = \min_{t''\in\mathbb{T}^{n-p}} \widetilde{B}(t'')$. We also choose

$$f_j \equiv 0, \quad j = p+1,\dots,n.$$

Given $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, for each $j = 1,\dots,p$ we obtain

$$\partial^\alpha \hat{f}_j(t,\xi) = \big[\partial^{(\alpha_1,\dots,\alpha_p)} g_j\big(t',\xi\big)\big] P_\xi\big(t''\big)e^{\xi(\widetilde{B}(t'')-M-ia_0''\cdot t'')}, \quad \xi \in \mathbb{Z}_+,$$

where P_ξ is a polynomial involving only powers of ξ (powers at most $|\alpha''| \doteq \alpha_{p+1} + \cdots + \alpha_n$) and derivatives of $\widetilde{B}(t'') - ia_0'' \cdot t''$ which are bounded on $\mathbb{T}^{n-p}$. Then, there exists a constant $C_\alpha > 0$ such that $|P_\xi(t'')| \leqslant C_\alpha|\xi|^{|\alpha''|}$ for all t''. Therefore

$$\big|\partial^\alpha \hat{f}_j(t,\xi)\big| \leqslant C_\alpha|\xi|^{|\alpha''|}\big|\partial^{(\alpha_1,\dots,\alpha_p)} g_j\big(t',\xi\big)\big|, \quad \xi \in \mathbb{Z}_+.$$

Analogously, we have the same inequality for $\xi \in \mathbb{Z}_-$. Since g_j are smooth functions it is possible to conclude by above inequality that f_j are smooth functions.

Finally, we define $f = \sum_{j=1}^n f_j\,dt_j$. It is easy to check that $f \in \mathbb{E}$.

Suppose that there exists $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ such that $\mathbb{L}u = f$. Then, if $u(t,x) = \sum_{\xi\in\mathbb{Z}} \hat{u}(t,\xi)e^{i\xi x}$ we have for each $\xi \in \mathbb{Z}$,

$$\begin{cases} \dfrac{\partial}{\partial t_j} w(t,\xi) + i\xi\big(a_{j0} + ib_j(t_j)\big)w(t,\xi) = e^{i\xi a_0''\cdot t''}\hat{f}_j(t,\xi), & j = 1,\dots,p, \\ \dfrac{\partial}{\partial t_j} w(t,\xi) - \xi b_j(t_j)w(t,\xi) = 0, & j = p+1,\dots,n, \end{cases}$$

where $w(t,\xi) \doteq \hat{u}(t,\xi)e^{i\xi a_0''\cdot t''}$. Thus, if $j = p+1,\dots,n$ we may write

$$\frac{\partial}{\partial t_j}\big(w(t,\xi)e^{-\xi(\widetilde{B}(t'')-M)}\big) = 0, \quad \text{if } \xi \geqslant 0,$$
$$\frac{\partial}{\partial t_j}\big(w(t,\xi)e^{-\xi(\widetilde{B}(t'')-m)}\big) = 0, \quad \text{if } \xi < 0.$$

Therefore,

$$w(t,\xi)e^{-\xi(\widetilde{B}(t'')-M)} \doteq \phi_\xi\big(t'\big), \quad \xi \geqslant 0,$$
$$w(t,\xi)e^{-\xi(\widetilde{B}(t'')-m)} \doteq \phi_\xi\big(t'\big), \quad \xi < 0.$$

Since $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ and

$$\left|\phi_\xi\left(t'\right)\right| = \left|w\left(t', t''_M, \xi\right)\right| = \left|\hat{u}\left(t', t''_M, \xi\right)\right|, \quad \xi \geqslant 0,$$
$$\left|\phi_\xi\left(t'\right)\right| = \left|w\left(t', t''_m, \xi\right)\right| = \left|\hat{u}\left(t', t''_m, \xi\right)\right|, \quad \xi < 0,$$

where $\widetilde{B}(t''_M) = M$ and $\widetilde{B}(t''_m) = m$ we have

$$\varphi\left(t', x\right) = \sum_{\xi \in \mathbb{Z}} \phi_\xi\left(t'\right) e^{i\xi x} \in \mathcal{D}'\left(\mathbb{T}^{p+1}_{(t',x)}\right).$$

On the other hand, if $j = 1, \ldots, p$ then

$$\frac{\partial}{\partial t_j}\phi_\xi\left(t'\right) + i\xi\left(a_{j0} + ib_j(t_j)\right)\phi_\xi\left(t'\right) = \hat{g}_j\left(t', \xi\right), \quad \xi \geqslant 0,$$
$$\frac{\partial}{\partial t_j}\phi_\xi\left(t'\right) + i\xi\left(a_{j0} + ib_j(t_j)\right)\phi_\xi\left(t'\right) = \hat{g}_j\left(t', \xi\right), \quad \xi < 0,$$

thus, we conclude that $\mathbb{L}^\# \varphi = g$, that is a contradiction. $\square$

5. Sufficiency in Proposition 2.4

If there is a function $b_j \not\equiv 0$ that does not change sign then $\mathbb{L}$ is globally solvable as in Proposition 2.3.

We may assume that $J = \{1, \ldots, m\}$, where $1 \leqslant m \leqslant n - 1$.

Assume now that $(a_{10}, \ldots, a_{m0}) \notin \mathbb{Q}^m$ is non-Liouville. Then, there exist a constant $C > 0$ and an integer $s > 1$ such that

$$\max_{1 \leqslant j \leqslant m} |\xi a_{j0} - \kappa_j| \geqslant \frac{C}{|\xi|^{s-1}}, \quad \forall (\kappa, \xi) \in \mathbb{Z}^m \times \mathbb{N}. \tag{5.1}$$

We will denote by (t', t'', x) the coordinates in the torus $\mathbb{T}^{n+1}$, where $t' = (t_1, \ldots, t_m)$ and $t'' = (t_{m+1}, \ldots, t_n)$.

Let $f(t, x) = \sum_{j=1}^n f_j(t, x)\, dt_j \in \mathbb{E}$. Consider the (t', x)-Fourier series as follows

$$u(t, x) = \sum_{(\kappa, \xi) \in \mathbb{Z}^m \times \mathbb{Z}} \hat{u}\left(t'', \kappa, \xi\right) e^{i(\kappa \cdot t' + \xi x)} \tag{5.2}$$

and

$$f_j(t, x) = \sum_{(\kappa, \xi) \in \mathbb{Z}^m \times \mathbb{Z}} \hat{f}_j\left(t'', \kappa, \xi\right) e^{i(\kappa \cdot t' + \xi x)}, \quad j = 1, \ldots, n, \tag{5.3}$$

where $\hat{u}(t'', \kappa, \xi)$ and $\hat{f}_j(t'', \kappa, \xi)$ denote the Fourier transform with respect to variables (t', x).

Substituting the formal series (5.2) and (5.3) in the equations $L_j u = f_j$, $j = 1, \ldots, m$, we have for each $(\kappa, \xi) \neq (0, 0)$,

$$i(\kappa_j + \xi a_{j0})\hat{u}\left(t'', \kappa, \xi\right) = \hat{f}_j\left(t'', \kappa, \xi\right), \quad j = 1, \ldots, m.$$

Also, from the compatibility conditions $L_j f_l = L_l f_j$, $j, l = 1, \ldots, m$, we obtain the following equations

$$(\kappa_j + \xi a_{j0})\hat{f}_l(t'', \kappa, \xi) = (\kappa_l + \xi a_{l0})\hat{f}_j(t'', \kappa, \xi), \quad j, l = 1, \ldots, m.$$

The preceding equations imply

$$\hat{u}(t'', \kappa, \xi) = \frac{1}{i(\kappa_N + \xi a_{N0})}\hat{f}_N(t'', \kappa, \xi), \quad (\kappa, \xi) \neq (0, 0), \tag{5.4}$$

where $1 \leqslant N \leqslant m$, $N = N(\xi)$ is such that

$$|\kappa_N + \xi a_{N0}| = \max_{1 \leqslant j \leqslant m} |\kappa_j + \xi a_{j0}|;$$

hence, $|\kappa_N + \xi a_{N0}| \neq 0$.

Since $\hat{f}(t'', 0, 0)$ is exact there is $v \in C^\infty(\mathbb{T}^{n-m}_{t''})$ such that $d_{t''} v = \hat{f}(\cdot, 0, 0)$. Thus, we choose $\hat{u}(t'', 0, 0) = v(t'')$.

Given $\alpha \in \mathbb{Z}^{n-m}_+$ and a positive integer L we obtain, from (5.4) and (5.1),

$$\begin{aligned}\left|\partial^\alpha \hat{u}(t'', \kappa, \xi)\right| &\leqslant \frac{1}{C}|\xi|^{s-1}\left|\partial^\alpha \hat{f}_N(t'', \kappa, \xi)\right| \\ &\leqslant C_\alpha \frac{1}{(1 + |(\kappa, \xi)|)^L}, \quad (\kappa, \xi) \neq (0, 0),\end{aligned}$$

since each f_j is a smooth function.

We conclude that

$$u(t, x) = \sum_{(\kappa, \xi) \in \mathbb{Z}^m \times \mathbb{Z}} \hat{u}(t'', \kappa, \xi) e^{i(\kappa \cdot t' + \xi x)} \in C^\infty(\mathbb{T}^{n+1}).$$

By construction u is a solution of

$$L_j u = f_j, \quad j = 1, \ldots, m.$$

Calculations similar to previous ones in this work show that u is also a solution to

$$L_j u = f_j, \quad j = m+1, \ldots, n.$$

We have thus proved that condition ii) implies global solvability.

Suppose now that iii) holds. Let $\mathcal{A} \doteq q_* \mathbb{Z}$ and $\mathcal{B} \doteq \mathbb{Z} \backslash \mathcal{A}$ and define

$$\begin{aligned}\mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1}) &\doteq \left\{u \in \mathcal{D}'(\mathbb{T}^{n+1}),\ u(t, x) = \sum_{\xi \in \mathcal{A}} \hat{u}(t, \xi) e^{i\xi x}\right\} \\ &= \left\{u \in \mathcal{D}'(\mathbb{T}^{n+1}),\ u(t, x) = \sum_{N \in \mathbb{Z}} \hat{u}(t, q_* N) e^{iq_* N x}\right\}.\end{aligned}$$

Let $\mathbb{L}_{\mathcal{A}}$ be the operator $\mathbb{L}$ acting on $\mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1})$. Similarly, we define $\mathcal{D}'_{\mathcal{B}}(\mathbb{T}^{n+1})$ and $\mathbb{L}_{\mathcal{B}}$. Then $\mathbb{L}$ is globally solvable if and only if $\mathbb{L}_{\mathcal{A}}$ and $\mathbb{L}_{\mathcal{B}}$ are globally solvable (see [3]).

Lemma 5.1. *$\mathbb{L}_{\mathcal{A}}$ is globally solvable.*

Proof. Since $q_* a_0 \in \mathbb{Z}^n$, we define

$$T : \mathcal{D}'_{\mathcal{A}}\big(\mathbb{T}^{n+1}\big) \longrightarrow \mathcal{D}'_{\mathcal{A}}\big(\mathbb{T}^{n+1}\big),$$
$$\sum_{\xi\in\mathbb{Z}} \hat{u}(t, q_*\xi) e^{iq_*\xi x} \longmapsto \sum_{\xi\in\mathbb{Z}} \hat{u}(t, q_*\xi) e^{-iq_*\xi a_0\cdot t} e^{iq_*\xi x}.$$

We note that T is an automorphism of $\mathcal{D}'_{\mathcal{A}}(\mathbb{T}^{n+1})$ and of $C^\infty_{\mathcal{A}}(\mathbb{T}^{n+1})$. Furthermore the following relation holds:

$$T^{-1}\mathbb{L}_{\mathcal{A}} T = \mathbb{L}_{0,\mathcal{A}} \doteq d_t + ib(t) \wedge \frac{\partial}{\partial x}.$$

Let B be a global primitive of b on $\mathbb{T}^n$. Since $\Omega_s = \{t \in \mathbb{T}^n,\ B(t) < s\}$ and $\Omega^s = \{t \in \mathbb{T}^n,\ B(t) > s\}$ are connected for every $s \in \mathbb{R}$ we have $\mathbb{L}_0$ globally solvable, hence $\mathbb{L}_{0,\mathcal{A}}$ is globally solvable. From $T^{-1}\mathbb{L}_{\mathcal{A}} T = \mathbb{L}_{0,\mathcal{A}}$ we obtain that $\mathbb{L}_{\mathcal{A}}$ is globally solvable.

If $q_J = q_* = 1$ then $\mathcal{A} = \mathbb{Z}$ and the proof is complete. □

Otherwise we have:

Lemma 5.2. *$\mathbb{L}_{\mathcal{B}}$ is globally solvable.*

Proof. Let $(\kappa, \xi) \in \mathbb{Z}^m \times \mathcal{B}$. Since q_* is the smallest natural such that $q_*(a_{10}, \dots, a_{m0}) \in \mathbb{Z}^m$ there exists $j \in \{1, \dots, m\}$ such that

$$\left| a_{j0} - \frac{\kappa_j}{\xi} \right| \geqslant \frac{1}{q_*|\xi|},$$

thus

$$\max_{\ell=1,\dots,m} \left| a_{\ell 0} - \frac{\kappa_\ell}{\xi} \right| \geqslant \left| a_{j0} - \frac{\kappa_j}{\xi} \right| \geqslant \frac{1}{q_*|\xi|}, \quad (\kappa, \xi) \in \mathbb{Z}^m \times \mathcal{B}.$$

Therefore, if the denominators $\xi \in \mathcal{B}$ then $a'_0 = (a_{10}, \dots, a_{m0})$ behaves as non-Liouville. The rest of the proof is analogous to the case where a'_0 is non-Liouville. □

6. Necessity in Proposition 2.4

We will suppose that $\mathbb{L}$ is globally solvable and none of the conditions i)–iii) holds. Again, as in Section 5 we will assume that $J = \{1, \dots, m\}$, then the global primitive of $\Pi^* b$ via the universal covering $\Pi : \mathbb{R}^n \to \mathbb{T}^n$ is of the form $B(t) = B(t'') = \sum_{j=m+1}^n B_j(t_j)$. Since each b_j, $j = m+1, \dots, n$, changes sign we may assume that the following conditions hold:

- ⋆ $b_{j0} \leqslant 0$, $j = m+1, \dots, n$.
- ⋆ The maximum of B over $[0, 2\pi]^{n-m}_{t''}$ is not attained at the boundary.
- ⋆ $B(0) = 0$.

Case 1. $a'_0 = (a_{10}, \dots, a_{m0}) \notin \mathbb{Q}^m$ is Liouville.

In this case we will consider two subcases, namely:

Case 1.1: $c_0'' = (c_{(m+1)0}, \dots, c_{n0}) \in \mathbb{Q}^{n-m}$;
Case 1.2: $c_0'' = (c_{(m+1)0}, \dots, c_{n0}) \notin \mathbb{Q}^{n-m}$.

Case 1.1: $c_0'' \in \mathbb{Q}^{n-m}$.

Since $a_0' \notin \mathbb{Q}^m$ is Liouville there are a constant $C > 0$ and a sequence $\{(\kappa_l, \xi_l)\}$ in $\mathbb{Z}^m \times \mathbb{Z}$, $\xi_l \geqslant 2$ such that

$$\max_{j=1,\dots,m} \left| a_{j0} - \frac{\kappa_l^{(j)}}{\xi_l} \right| \leqslant \frac{C}{(\xi_l)^l}, \quad \forall l \in \mathbb{N},$$

where we may assume that $\xi_l < \xi_{l+1}$ for all l.

Let $\mathcal{A} = \eta\mathbb{Z}$ where $\eta \in \mathbb{N}$ is such that $\eta c_0'' \in \mathbb{Z}^{n-m}$. For each $l \in \mathbb{N}$ we claim that it is possible to obtain $\xi_l, \kappa_l^{(j)} \in \mathcal{A}$, $j = 1, \dots, m$, in the above sequence $\{(\kappa_l, \xi_l)\}$. Indeed, if $\eta = 1$ the conclusion is obvious. If $\eta > 1$ we have that $\eta \leqslant \xi_\eta \leqslant \xi_{\eta l} \leqslant (\xi_{\eta l})^{\eta-1}$ for all $l \in \mathbb{N}$. Consider now the subsequence $\{(\kappa_{\eta l}, \xi_{\eta l})\}$, then

$$\max_{j=1,\dots,m} \left| a_{j0} - \frac{\eta\kappa_{\eta l}^{(j)}}{\eta\xi_{\eta l}} \right| \leqslant C(\xi_{\eta l})^{-\eta l} = C \underbrace{\left(\eta^l \xi_{\eta l}^{-(\eta-1)l}\right)}_{\leqslant 1} (\eta\xi_{\eta l})^{-l} \leqslant C(\eta\xi_{\eta l})^{-l},$$

for all $l \in \mathbb{N}$. Now, it suffices to choose $(\tilde{\kappa}_l, \tilde{\xi}_l) = (\eta\kappa_{\eta l}, \eta\xi_{\eta l})$, $l \in \mathbb{N}$.

Let $\mathbb{L}^\#$ be the involutive system generated by the vector fields $L_1, L_2, \dots, L_m$. The preceding argument and the result in [7] imply that $\mathbb{L}^\#_{\mathcal{A}}$ is not globally solvable on $\mathbb{T}^{m+1}$.

We conclude that $\mathbb{L}_{\mathcal{A}}$ is not globally solvable in view of the following result, the proof of which will be omitted since it is a simple adaptation of Lemma 4.4.

Lemma 6.1. *Let $1 \leqslant p \leqslant n$. Consider $\mathcal{A} = \eta\mathbb{Z}$ where $\eta \in \mathbb{N}$ and $\mathbb{L}^\#$ the involutive system generated by $L_1, \dots, L_p$. If $\mathbb{L}^\#_{\mathcal{A}}$ is not globally solvable on $\mathbb{T}^{p+1}$ and $\eta(c_{(p+1)0}, \dots, c_{n0}) \in \mathbb{Z}^{n-p}$ then $\mathbb{L}_{\mathcal{A}}$ is not globally solvable on $\mathbb{T}^{n+1}$.*

Case 1.2: $c_0'' \notin \mathbb{Q}^{n-m}$.

Assume first that $\{c_{(m+1)0}, \dots, c_{n0}\} \cap \mathbb{Q} = \emptyset$. In this case the nonsolvability of $\mathbb{L}$ will be given by Propositions 6.2–6.5.

As in Section 4 we will use the equality

$$L_n \dots L_2 L_1 u = L_{\sigma(n)} \dots L_{\sigma(2)} L_{\sigma(1)} u, \quad \sigma \in \mathcal{S}_n, \tag{6.1}$$

to find smooth functions $f_1, \dots, f_n$ such that $f = \sum f_j dt_j \in \mathbb{E}$ and $\mathbb{L}u = f$ has no solution u in $\mathcal{D}'(\mathbb{T}^{n+1})$. Using the formal series

$$u(t,x) = \sum_{(\kappa,\xi) \in \mathbb{Z}^m \times \mathbb{Z}} \hat{u}\left(t'', \kappa, \xi\right) e^{i(\kappa \cdot t' + \xi x)}, \tag{6.2}$$

$$h(t,x) = \sum_{(\kappa,\xi) \in \mathbb{Z}^m \times \mathbb{Z}} \hat{h}\left(t'', \kappa, \xi\right) e^{i(\kappa \cdot t' + \xi x)}, \tag{6.3}$$

in the equation $h \doteq L_n \dots L_2 L_1 u$, we obtain for each $(\kappa, \xi) \in \mathbb{Z}^m \times \mathbb{Z}$,

$$\hat{h}(t'', \kappa, \xi) = \left(\prod_{j=m+1}^{n} \left(\partial_{t_j} + i\xi c_j(t_j)\right) \prod_{j=1}^{m} i(\kappa_j + \xi a_{j0}) \right) \hat{u}(t'', \kappa, \xi), \tag{6.4}$$

where $c_j(t_j) = a_{j0} + ib_j(t_j)$, $j = m+1, \dots, n$.

Since $a_0' \notin \mathbb{Q}^m$ is Liouville it follows that there are a constant $C > 0$ and a sequence $\{(\kappa_l, \xi_l)\}$ in $\mathbb{Z}^m \times \mathbb{Z}$, $\xi_l \geqslant 2$ such that

$$\left|\kappa_l + \xi_l a_0'\right| \doteq \max_{j=1,\dots,m} \left|\kappa_l^{(j)} + \xi_l a_{j0}\right| \leqslant \frac{C}{|(\kappa_l, \xi_l)|^l}, \quad \forall l \in \mathbb{N}. \tag{6.5}$$

Therefore, for each $l \in \mathbb{N}$ Eq. (6.4) has exactly one solution given by

$$\hat{u}(t'', \kappa_l, \xi_l) = \delta_l \int_{[0,2\pi]^{n-m}} e^{i\xi_l(\mathcal{C}(t''-s'') - \mathcal{C}(t''))} \hat{h}(t'' - s'', \kappa_l, \xi_l)\, ds'', \tag{6.6}$$

where

$$\delta_l = \prod_{j=1}^{m} \left[i\left(\kappa_l^{(j)} + \xi_l a_{j0}\right)\right]^{-1} \prod_{j=m+1}^{n} \left[1 - e^{-i2\pi\xi_l c_{j0}}\right]^{-1}$$

and $\mathcal{C}(t'') = a_0'' \cdot t'' + iB(t'')$.

Let $v(t'', \kappa_l, \xi_l)$ be given by the right-hand side of (6.6). For each $l \in \mathbb{N}$ we define

$$\hat{f}_j(t'', \kappa_l, \xi_l) \doteq \begin{cases} i(\kappa_l^{(j)} + \xi_l a_{j0}) v(t'', \kappa_l, \xi_l) & \text{if } j = 1, \dots, m, \\ (\partial/\partial t_j + i\xi_l c_j(t_j)) v(t'', \kappa_l, \xi_l) & \text{if } j = m+1, \dots, n, \end{cases}$$

thus for $j = 1, \dots, m$ we have

$$\hat{f}_j(t'', \kappa_l, \xi_l) = i\left(\kappa_l^{(j)} + \xi_l a_{j0}\right)\delta_l \int_{[0,2\pi]^{n-m}} e^{i\xi_l(\mathcal{C}(t''-s'') - \mathcal{C}(t''))} \hat{h}(t'' - s'', \kappa_l, \xi_l)\, ds'', \tag{6.7}$$

and for $j = m+1, \dots, n$,

$$\hat{f}_j(t'', \kappa_l, \xi_l) = \frac{\delta_l}{d_{j\xi_l}} \int_{[0,2\pi]^{n-m-1}} e^{i\xi_l(\mathcal{C}(t''-s''+s_j e_j) - \mathcal{C}(t''))} \hat{h}(t'' - s'' + s_j e_j, \kappa_l, \xi_l)\, ds''_{(j)}, \tag{6.8}$$

where $d_{j\xi_l} = (1 - e^{-i2\pi\xi_l c_{j0}})^{-1}$ and $ds''_{(j)} = ds_{m+1} \dots ds_{j-1}\, ds_{j+1} \dots ds_n$.

Let

$$f_j(t, x) = \sum_{l \in \mathbb{N}} \hat{f}_j(t'', \kappa_l, \xi_l) e^{i(\kappa_l \cdot t' + \xi_l x)}, \quad j = 1, \dots, n.$$

In Propositions 6.3 and 6.4 we prove that $f_1, \dots, f_n$ are smooth functions for a convenient choice of function h. It follows from the definition of $\hat{f}_j$ that $f(t,x) = \sum_{j=1}^{n} f_j(t,x)\, dt_j \in \mathbb{E}$. Also, if there existed u such that $\mathbb{L}u = f$ then the coefficients would have to verify $\hat{u} = v$.

Construction of the function h: We will denote by M the maximum of $B = B(t'')$ over $[0, 2\pi]^{n-m}_{t''}$ and by M_j, $j = m+1, \dots, n$ the maximum of B over $[0, 2\pi]^{n-m}_{t''} \cap \{t_j = 0\}$. Thus, if $\widetilde{M}$ is the largest M_j we have that $0 \leqslant \widetilde{M} < M$ by the assumptions in the beginning of this section.

We define now the following sets

$$\mathcal{F}_1 = \left\{l \in \mathbb{N};\ e^{\xi_l \frac{M-\widetilde{M}}{2}} \left|\kappa_l + \xi_l a'_0\right| \leqslant 1\right\},$$

$$\mathcal{F}_2 = \left\{l \in \mathbb{N};\ e^{\xi_l \frac{M-\widetilde{M}}{2}} \left|\kappa_l + \xi_l a'_0\right| > 1\right\}.$$

Let $(\gamma_l)_{l\in\mathbb{N}}$ be the sequence given by

$$\gamma_l = \begin{cases} e^{\lambda \xi_l} & \text{if } l \in \mathcal{F}_1, \\ |\kappa_l + \xi_l a'_0|^{-1/2} & \text{if } l \in \mathcal{F}_2, \end{cases}$$

where $0 < \lambda < (M - \widetilde{M})/2$ is a constant. Note that the sequence $(\gamma_l)_{l\in\mathbb{N}}$ does not correspond to any Fourier transform of a periodic distribution (see (6.5)).

Let $\chi_\delta \in C_c^\infty(\mathbb{R}^{n-m})$ be a function such that

$$\chi_\delta\big(t''\big) = \begin{cases} 1 & \text{if } t'' \in (-\delta, \delta)^{n-m}, \\ 0 & \text{if } t'' \in \mathbb{R}^{n-m} \backslash (-2\delta, 2\delta)^{n-m} \end{cases}$$

and $0 \leqslant \chi_\delta(t'') \leqslant 1$, for all t'', where $0 < \delta < \pi/2$.

We define

$$p\big(t''\big) = \sum_{\nu\in\mathbb{Z}^{n-m}} a''_0 \cdot \big(t''_* - t'' - 2\pi\nu\big)\chi_\delta\big(t'' + 2\pi\nu\big),$$

$$q\big(t''\big) = 1 + \sum_{\nu\in\mathbb{Z}^{n-m}} \big(\big|t'' + 2\pi\nu\big|^2 - 1\big)\chi_\delta\big(t'' + 2\pi\nu\big),$$

$$r\big(t''\big) = \sum_{\nu\in\mathbb{Z}^{n-m}} B\big(t'' + 2\pi\nu\big)\chi_\delta\big(t'' + 2\pi\nu\big),$$

where $t''_* \in (0, 2\pi)^{n-m}$ is such that $B(t''_*) = M$. Then p, q and r are periodic smooth functions and the following properties hold:

- $q(t'') \geqslant 0$, $\forall t'' \in \mathbb{R}^{n-m}$; $q(t'') = 0$ if and only if $t'' = 2\pi\nu$, $\nu \in \mathbb{Z}^{n-m}$.
- $r(2\pi\nu) = 0$, $\forall \nu \in \mathbb{Z}^{n-m}$.
- If $|t''| < \delta$ then $p(t'') = a''_0 \cdot (t''_* - t'')$, $q(t'') = |t''|^2$ and $r(t'') = B(t'')$.

Finally, we define

$$h(t, x) = \sum_{l\in\mathbb{N}} \hat{h}\big(t'', \kappa_l, \xi_l\big)e^{i(\kappa_l\cdot t' + \xi_l x)}$$

where

$$\hat{h}\big(t'', \kappa_l, \xi_l\big) = \frac{\gamma_l}{\delta_l} e^{\xi_l(-Kq(t'') + r(t'') - M + ip(t''))} \tag{6.9}$$

and $K > 0$ will be chosen later on.

Proposition 6.2. $h \in C^\infty(\mathbb{T}^{n+1})$.

Proof. If $\phi(t'') = -Kq(t'') + r(t'') - M$ then $\phi(2\nu\pi) = -M$, $\forall \nu \in \mathbb{Z}^{n-m}$. Since ϕ is continuous we have $\phi(t'') < -M + \lambda$ in a neighborhood U of the vertices of cube $[0, 2\pi]^{n-m}$.

Let $\sigma \doteq \min\{q(t'');\ t'' \in [0, 2\pi]^{n-m} \setminus U\}$, thus $\sigma > 0$. Now, if necessary, we take a larger $K > 0$ to obtain $\phi(t'') < -M + \lambda$ on the compact $[0, 2\pi]^{n-m} \setminus U$. Thus $\phi(t'') < -M + \lambda,$ for all $t'' \in [0, 2\pi]^{n-m}$.

Furthermore, there is a constant $C > 0$ such that

$$\frac{1}{|\delta_l|} \leqslant C\left|\kappa_l + \xi_l a_0'\right|^m, \quad \forall l \in \mathbb{N},$$

hence, for each $l \in \mathbb{N}$, we obtain

$$\begin{aligned}\left|\hat{h}\left(t'', \kappa_l, \xi_l\right)\right| &\leqslant C\gamma_l\left|\kappa_l + \xi_l a_0'\right|^m e^{-\xi_l(M-\lambda)} \\ &= \begin{cases} C|\kappa_l + \xi_l a_0'|^m e^{-\xi_l(M-2\lambda)} & \text{if } l \in \mathcal{F}_1, \\ C|\kappa_l + \xi_l a_0'|^{(2m-1)/2} e^{-\xi_l(M-\lambda)} & \text{if } l \in \mathcal{F}_2. \end{cases}\end{aligned}$$

One can show that all derivatives of $\hat{h}$ satisfy similar estimates; hence, $h \in C^\infty(\mathbb{T}^{n+1})$. $\square$

Substituting the coefficients (6.9) in Eqs. (6.6)–(6.8) we obtain, for each $l \in \mathbb{N}$, the following equations

$$\hat{f}_j\left(t'', \kappa_l, \xi_l\right) = i\left(\kappa_l^{(j)} + \xi_l a_{j0}\right)\gamma_l \int\limits_{[0,2\pi]^{n-m}} e^{\xi_l\varphi_1(t'',s'')} e^{i\xi_l\varphi_2(t'',s'')}\, ds'', \tag{6.10}$$

for $j = 1, \ldots, m$,

$$\hat{f}_j\left(t'', \kappa_l, \xi_l\right) = \frac{\gamma_l}{d_{j\xi_l}} \int\limits_{[0,2\pi]^{n-m-1}} e^{\xi_l\varphi_1(t'',s''-s_je_j)} e^{i\xi_l\varphi_2(t'',s''-s_je_j)}\, ds''_{(j)}, \tag{6.11}$$

for $j = m+1, \ldots, n$, and

$$\hat{u}\left(t'', \kappa_l, \xi_l\right) = \gamma_l \int\limits_{[0,2\pi]^{n-m}} e^{\xi_l\varphi_1(t'',s'')} e^{i\xi_l\varphi_2(t'',s'')}\, ds'', \tag{6.12}$$

where $\varphi_1(t'', s'') = B(t'') - B(t'' - s'') - Kq(t'' - s'') + r(t'' - s'') - M$ and $\varphi_2(t'', s'') = p(t'' - s'') - a_0'' \cdot s''$ are real valued functions.

Proposition 6.3. *$f_j \in C^\infty(\mathbb{T}^{n+1})$ for $j = 1, \ldots, m$.*

Proof. We first show that $\varphi_1(t'', s'') \leqslant 0$, for all $t'', s'' \in [0, 2\pi]^{n-m}$. Denote by F the boundary of the cube $[0, 2\pi]^{n-m}$. Since $B < M$ on F there exists $0 < \delta_0 \leqslant \delta$ such that $B \leqslant M$ on

$$U \doteq \left\{\tau'' \in \mathbb{R}^{n-m},\ \operatorname{dist}\left(\tau'', \mathrm{F}\right) < \delta_0\right\}.$$

Now for each $2\pi\nu,\ \nu \in \mathbb{Z}^{n-m}$, we consider the open ball of radius $\delta_0 > 0$ centered at $2\pi\nu$. Let V be the (disjoint) union of these balls.

Given $t'', s'' \in [0, 2\pi]^{n-m}$ we have $t'' - s'' \in [-2\pi, 2\pi]^{n-m}$.

Suppose first that $t'' - s'' \in V$; then we have $|(t'' - s'') - 2\pi\nu| < \delta_0$ for some $2\pi\nu$, $\nu \in \mathbb{Z}^{n-m}$, and

$$\begin{aligned}\varphi_1(t'', s'') &= B(t'') - B(t'' - s'') - Kq(t'' - s'') + r(t'' - s'' - 2\pi\nu) - M \\ &= B(t'') - B(t'' - s'') - Kq(t'' - s'') + B(t'' - s'' - 2\pi\nu) - M \\ &= B(t'') - M - Kq(t'' - s'') - 2\pi\nu \cdot b_0'' \\ &= B(t'' - 2\pi\nu) - M - Kq(t'' - s'') \leqslant B(t'' - 2\pi\nu) - M,\end{aligned}$$

hence, if $\nu = 0$ then $\varphi_1(t'', s'') \leqslant 0$. If $\nu \neq 0$ then $(t'' - 2\pi\nu) \notin (0, 2\pi)^{n-m}$ and

$$\operatorname{dist}(t'' - 2\pi\nu, \mathrm{F}) \leqslant \operatorname{dist}(t'' - 2\pi\nu, s'') = \operatorname{dist}(t'' - s'', 2\pi\nu) < \delta_0,$$

which implies that $t'' - 2\pi\nu \in U$, and $\varphi_1(t'', s'') \leqslant 0$.

Suppose now that $t'' - s'' \notin V$; then

$$\begin{aligned}\varphi_1(t'', s'') &\leqslant -B(t'' - s'') - Kq(t'' - s'') + r(t'' - s'') \\ &\leqslant C - K\sigma,\end{aligned}$$

where $C = \max\{r(\tau'') - B(\tau''),\ \tau'' \in [-2\pi, 2\pi]^{n-m} \setminus V\}$ and

$$0 < \sigma \doteq \min\{q(\tau''),\ \tau'' \in [-2\pi, 2\pi]^{n-m} \setminus V\}.$$

Now, if necessary, we take a larger $K > 0$ such that $C \leqslant K\sigma$.

We obtain from (6.10)

$$\begin{aligned}|\hat{f}_j(t'', \kappa_l, \xi_l)| &\leqslant (2\pi)^{n-m}\gamma_l|\kappa_l + \xi_l a_0'| \\ &\leqslant (2\pi)^{n-m}\begin{cases} e^{-\xi_l(\frac{M-\widetilde{M}}{2}-\lambda)} & \text{if } l \in \mathcal{F}_1, \\ |\kappa_l + \xi_l a_0'|^{1/2} & \text{if } l \in \mathcal{F}_2.\end{cases}\end{aligned}$$

As before we conclude that $f_1, \ldots, f_m \in C^\infty(\mathbb{T}^{n+1})$. $\square$

Proposition 6.4. *$f_j \in C^\infty(\mathbb{T}^{n+1})$ for $j = m+1, \ldots, n$.*

Proof. We will show that $\varphi_1(t'', s'' - s_je_j) \leqslant (\widetilde{M} - M)/2$, for all $t'', s'' \in [0, 2\pi]^{n-m}$. We begin by defining the set

$$\mathcal{Q}_j = \{(t'', s'' - s_je_j) \in [0, 2\pi]^{n-m} \times [0, 2\pi]^{n-m-1};\ q(t'' - s'' + s_je_j) = 0\}.$$

If $(t'', s'' - s_je_j) \in \mathcal{Q}_j$ then $t'' - s'' + s_je_j = 2\pi\nu$, $\nu \in \mathbb{Z}^{n-m}$. Therefore

$$\begin{aligned}\varphi_1(t'', s'' - s_je_j) &= B(s'' - s_je_j + 2\pi\nu) - B(2\pi\nu) + r(2\pi\nu) - M \\ &= B(s'' - s_je_j) - M \\ &\leqslant M_j - M < \frac{\widetilde{M} - M}{2}, \quad (t'', s'' - s_je_j) \in \mathcal{Q}_j.\end{aligned}$$

Since $\mathcal{Q}_j$ is closed there exists a neighborhood U of $\mathcal{Q}_j$ such that

$$\varphi_1\big(t'', s'' - s_j e_j\big) < \frac{\widetilde{M} - M}{2}, \quad \big(t'', s'' - s_j e_j\big) \in U.$$

If $\sigma \doteq \min\{q(t'' - s'' + s_j e_j);\ (t'', s'' - s_j e_j) \in \mathcal{K}\}$ where $\mathcal{K}$ is the compact set $([0, 2\pi]^{n-m} \times [0, 2\pi]^{n-m-1}) \setminus U$, then $\sigma > 0$.

Now, if necessary, we take a larger $K > 0$ such that

$$\varphi_1\big(t'', s'' - s_j e_j\big) \leqslant \frac{\widetilde{M} - M}{2}, \quad \big(t'', s'' - s_j e_j\big) \in \mathcal{K}.$$

By (6.11) we have

$$\big|\hat{f}_j\big(t'', \kappa_l, \xi_l\big)\big| \leqslant C\gamma_l e^{-\xi_l \frac{M-\widetilde{M}}{2}} \leqslant \begin{cases} Ce^{-\xi_l(\frac{M-\widetilde{M}}{2}-\lambda)} & \text{if } l \in \mathcal{F}_1, \\ Ce^{-\xi_l(M-\widetilde{M})/4} & \text{if } l \in \mathcal{F}_2, \end{cases}$$

where $C > 0$ does not depend on t'' and (κ_l, ξ_l). The rest of the proof goes through as before. $\square$

Proposition 6.5. *For $K > 0$ sufficiently large there is no distribution $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ whose sequence of (t', x)-Fourier coefficients is the sequence* (6.12)*.*

Proof. Let $t''_* \in (0, 2\pi)^{n-m}$ such that $B(t''_*) = M$. We write

$$\hat{u}\big(t''_*, \kappa_l, \xi_l\big) = \gamma_l(I_l + J_l),$$

where

$$I_l = \int\limits_{|t''_* - s''| < \delta} e^{\xi_l \varphi_1(t''_*, s'')} e^{i\xi_l \varphi_2(t''_* - s'')}\, ds'',$$

$$J_l = \int\limits_{|t''_* - s''| \geqslant \delta} e^{\xi_l \varphi_1(t''_*, s'')} e^{i\xi_l \varphi_2(t''_* - s'')}\, ds''.$$

If we make the change of variables $K\xi_l(t''_* - s'') = \sigma$ we obtain

$$\begin{aligned} I_l &= \int\limits_{|t''_* - s''| < \delta} e^{\xi_l \varphi_1(t''_*, s'')} e^{i\xi_l \varphi_2(t''_* - s'')}\, ds'' \\ &= \int\limits_{|t''_* - s''| < \delta} e^{-K\xi_l |t''_* - s''|^2}\, ds'' \\ &= \frac{1}{(K\xi_l)^{(n-m)/2}} \int\limits_{|\sigma| < \delta\sqrt{K\xi_l}} e^{-|\sigma|^2}\, d\sigma \\ &= \frac{1}{(K\xi_l)^{(n-m)/2}} \mu_l, \end{aligned}$$

where $\mu_l \doteq \int_{|\sigma| < \delta\sqrt{K\xi_l}} e^{-|\sigma|^2}\, d\sigma$ is an increasing sequence of positive real numbers.

On the other hand, on the cube $Q^* = t^* - [0, 2\pi]^{n-m}$ we have $0 < \sigma = \min\{q(\mu);\ \mu \in Q^* \text{ and } |\mu| \geqslant \delta\}$.

Now, if necessary, we take a larger $K > 0$ such that

$$\varphi\big(t''_*, s''\big) \leqslant -K\sigma + r\big(t''_* - s''\big) - B\big(t''_* - s''\big) < -\lambda, \quad \forall s'' \in [0, 2\pi]^{n-m},$$

hence,

$$|J_l| \leqslant \int\limits_{|t''_* - s''| \geqslant \delta} e^{\xi_l \varphi_1(t''_*, s'')}\, ds'' \leqslant (2\pi)^{n-m} e^{-\xi_l \lambda}.$$

Therefore, we have

$$\begin{aligned} \big|\hat{u}\big(t''_*, \kappa_l, \xi_l\big)\big| &\geqslant \gamma_l\big(|I_l| - |J_l|\big) \\ &\geqslant \gamma_l\left(\frac{1}{(K\xi_l)^{(n-m)/2}}\mu_l - (2\pi)^{n-m} e^{-\xi_l \lambda}\right). \end{aligned}$$

Now, we take $N \in \mathbb{N}$ such that $\frac{e^{\xi_l \lambda}\mu_1}{(2\pi)^{n-m}(K\xi_l)^{(n-m)/2}} \geqslant 1$ for all $\xi_l > N$. Thus, by the preceding inequality we obtain

$$\begin{aligned} \big|\hat{u}\big(t''_*, \kappa_l, \xi_l\big)\big| &\geqslant \gamma_l\left(\frac{1}{(K\xi_l)^{(n-m)/2}}\mu_l - (2\pi)^{n-m} e^{-\xi_l \lambda}\right) \\ &\geqslant \gamma_l\left(\frac{1}{(K\xi_l)^{(n-m)/2}}\mu_l - \frac{1}{(K\xi_l)^{(n-m)/2}}\mu_1\right) \\ &= \gamma_l(\mu_l - \mu_1)\frac{1}{(K\xi_l)^{(n-m)/2}} \\ &\geqslant \frac{(\mu_l - \mu_1)\gamma_l}{(K\xi_l)^{(n-m)/2}}, \quad \forall \xi_l > N. \end{aligned}$$

Therefore, the sequence $\{\hat{u}(t''_*, \kappa_l, \xi_l)\}$ does not correspond to any distribution u in $\mathcal{D}'(\mathbb{T}^{n+1})$.

We now address the case when $c''_0 \notin \mathbb{Q}^{n-m}$, but $\{c_{(m+1)0}, \ldots, c_{n0}\} \cap \mathbb{Q} \neq \emptyset$.

We may assume that $c_{(m+1)0}, \ldots, c_{p0} \notin \mathbb{Q}$ and $c_{(p+1)0}, \ldots, c_{n0} \in \mathbb{Q}$ where $m + 1 \leqslant p < n$.

Let $\mathbb{L}^\#$ be the system generated by the vector fields $L_1, \ldots, L_p$ and $\mathcal{A} = \eta\mathbb{Z}$ where $\eta \in \mathbb{N}$ is such that $\eta(c_{(p+1)0}, \ldots, c_{n0}) \in \mathbb{Z}^{n-p}$.

For each $l \in \mathbb{N}$ we may assume that $\xi_l, \kappa_l^{(j)} \in \mathcal{A}$, $j = 1, \ldots, m$ in the sequence $\{(\kappa_l, \xi_l)\}$ considered previously, hence $\mathbb{L}^\#_{\mathcal{A}}$ is not globally solvable on $\mathbb{T}^{n-p+1}$ by the first part in Case 1.2. Finally, by Lemma 6.1 we conclude that $\mathbb{L}_{\mathcal{A}}$ is not globally solvable on $\mathbb{T}^{n+1}$ which implies that $\mathbb{L}$ is not globally solvable on $\mathbb{T}^{n+1}$. □

Case 2. $a'_0 = (a_{10}, \ldots, a_{m0}) \in \mathbb{Q}^m$.

Assume first that $c''_0 = (c_{(m+1)0}, \ldots, c_{n0}) \notin \mathbb{Q}^{n-m}$, that is, either b is not exact or $a''_0 \notin \mathbb{Q}^{n-m}$. We may assume that $c_{(m+1)0}, \ldots, c_{p0} \in \mathbb{Q}$ and $c_{(p+1)0}, \ldots, c_{n0} \notin \mathbb{Q}$ where $m \leqslant p < n$ ($p = m$ means that $c_{(m+1)0}, \ldots, c_{n0} \notin \mathbb{Q}$). Let $\mathcal{A} = \eta\mathbb{Z}$ where $\eta \in \mathbb{N}$ is such that $\eta(c_{10}, \ldots, c_{p0}) \in \mathbb{Z}^p$ and consider the involutive system $\mathbb{L}^\#$ generated by the vector fields $L_{p+1}, \ldots, L_n$.

Recall that in the construction of $f \in \mathbb{E}$ in Section 4 we considered only $\xi \in \mathbb{Z} \setminus \mathcal{G}$, where $\mathcal{G}$ is given by (4.2). Since $c_{(p+1)0}, \ldots, c_{n0} \notin \mathbb{Q}$ (hence $\mathcal{G} = \emptyset$) the same construction shows that $\mathbb{L}^\#_{\mathcal{A}}$ is not globally solvable using only $\xi \in \mathcal{A}$. Thus, by Lemma 6.1 we conclude that $\mathbb{L}_{\mathcal{A}}$ is not globally solvable on $\mathbb{T}^{n+1}$.

Assume now that $c_0'' = (c_{(m+1)0}, \ldots, c_{n0}) \in \mathbb{Q}^{n-m}$ (that is, b is exact and $a_0'' \in \mathbb{Q}^{n-m}$) and $q_J < q_*$. We may assume that $q_J a_{10}, \ldots, q_J a_{p,0} \in \mathbb{Z}$ and $q_J a_{(p+1)0}, \ldots, q_J a_{n0} \notin \mathbb{Z}$ where $m \leqslant p < n$.

We write $a_{j0} = r_j/s_j$, $j = p+1, \ldots, n$ and since $\mathcal{G} = \bigcup_{j=p+1}^{n} s_j\mathbb{Z}$, $q_J\mathbb{Z}_+ \setminus \mathcal{G}$ has infinitely many elements. Therefore, as in Section 4 we have that $\mathbb{L}^{\#}_{\mathcal{A}}$ is not globally solvable on $\mathbb{T}^{n-p+1}_{(t_{p+1},\ldots,t_n,x)}$ where $\mathbb{L}^{\#}$ is the involutive system generated by $L_{p+1}, \ldots, L_n$ and $\mathcal{A} = q_J\mathbb{Z}$. Thus, by Lemma 6.1 we have $\mathbb{L}_{\mathcal{A}}$ not globally solvable on $\mathbb{T}^{n+1}$.

Finally, if $a_0 \in \mathbb{Q}^n$, b is exact, $q_J = q_*$ and the global primitive $B : \mathbb{T}^n \to \mathbb{R}$ of b has a disconnected sublevel or superlevel on $\mathbb{T}^n$ then $\mathbb{L}$ is not globally solvable. Indeed, by Lemma 5.1 we have that $\mathbb{L}_{\mathcal{A}}$ is globally solvable if and only if $\mathbb{L}_{0,\mathcal{A}}$ is globally solvable, where $\mathcal{A} = q_*\mathbb{Z}$ and $\mathbb{L}_0 = d_t + ib(t) \wedge \frac{\partial}{\partial x}$. Since B has a disconnected sublevel or superlevel, we have $\mathbb{L}_{0,\mathcal{A}}$ is not globally solvable by [9].

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